Quantization and integrability of discrete system

1992 J. Phys. A: Math. Gen. 25 6419

(http://iopscience.iop.org/0305-4470/25/23/035)

J. Phys. A: Math. Gen. 25 (1992) 6419–6427. Printed in the UK

Quantization and integrability of discrete systems

B Grammaticos†, A Ramani‡, V G Papageorgiou§ and F W Nijhoff§

† LPN, Université Paris VII, Tour 24-14, 5ème étage, 75251 Paris, France
‡ CPT, Ecole Polytechnique, 91128 Palaiseau, France
§ Department of Mathematics and Computer Science and Institute of Nonlinear Studies, Clarkson University, Potsdam, NY 13699-5815, USA

Received 1 April 1992

Abstract. The question of quantization of two-dimensional mappings representing the discrete Painlevé equations and their autonomous limits is examined. We show that for all these mappings it is possible to find a consistent quantization scheme, inspired from the commutation relations encountered in quantum groups. In the autonomous case we show that the classical invariant survives after the quantization, provided one introduces adequate quantum corrections in both the mapping and the invariant. For the discrete Painlevé equations themselves the integrability constraints are so stringent that they suffice even for the quantized case. In all the known cases the classical Lax pairs can be transcribed as quantal ones requiring only a (straightforward) choice of ordering for some of them.

1. Introduction

Modelling of physical systems proceeds often through discretization. Instead of considering continuous space and time variables one discretizes both; then the equations of motion become *difference* equations. This approach is interesting on several levels. Discrete systems lend themselves, by construction, to simulations: the lattice- or mapping-type equations of motion naturally provide integration schemes. Moreover space and time are treated on an equal footing. Lattice models are fundamental in the sense that they contain whole families of continuous systems that are obtained through the appropriate continuous limits.

The study of the integrability of discrete systems has, curiously, been neglected until very recently. While discrete systems have been used extensively for the understanding of chaos and its mechanisms, their integrability has barely been touched upon. The situation is now rapidly changing [1]. From the picture that emerges from recent studies one can assert that all the types of 'continuous' integrability have their 'discrete' counterpart. In [2], Quispel and collaborators have exhibited a large family of second-order mappings that possess an integral of motion and are the discrete analogues of elliptic functions. Systems linearizable through the discrete equivalent of Cole–Hopf transformations have been studied in [3]. But the most important result has been the discovery of systems that possess Lax pairs [4, 5]. These discrete equations are obtained as a compatibility condition for a linear system, through the discrete analogue of the Zakharov–Shabat procedure.

Several multidimensional integrable lattices but also one-dimensional, non-autonomous mappings have been obtained in this way. The latter turned out to be the discrete equivalents to the Painlevé equations. The first examples of such mappings d-P_I, d-P_{II} have been discovered in connection with two-dimensional quantum gravity [6], as well as similarity reductions of differential difference [4] or partial difference [5] equations, while the higher Painlevé equations, P_{III} to P_V, were discovered using the discrete analogue of the Painlevé method: the singularity confinement procedure introduced in [7]. To date Lax pairs are known for d-P_I of which three different forms exist, d-P_{II} and d-P_{III} [8]. The Lax pair in the present context† must be thought of as a pair of matrices, say L_n and M_n, depending on a spectral parameter, h. The compatibility condition reads, for example: $h\,\mathrm{d}M_n/\mathrm{d}h = L_{n+1}M_n - M_nL_n$. One interesting way to interpret this relation is, according to Novikov [9], as a quantization condition for the spectral curve. Thus the de-autonomization process is, in some formal sense, a first kind of quantization.

Another, more direct, kind of quantization is that which operates on the variables of the discrete system itself. Instead of c-numbers, the mapping variables become non-commuting operators. Recent studies have been devoted to this subject. In [10] the quantum integrability of a family of (classically) integrable lattices has been related to the existence of a well-defined quantum Yang–Baxter structure, which provides a complete set of commuting operators. A more direct approach to quantum integrability has been used in [11] concerning a mapping of the Quispel family. Starting from a restricted parametrization of the latter and postulating that the mapping variables (x, y) are operators satisfying canonical relations $[x, y] = \mathrm{i}\hbar$, it has been shown that it is possible to choose the right ordering for the classical invariant for it to stay invariant even in the quantum case.

What has not been attempted before is to combine the two 'quantizations': deautonomization and real-space quantization. Since the first procedure, for second-order mappings (starting from the Quispel family), leads to discrete Painlevé equations, one expects with this method to find their quantum versions.

The present study is devoted to the problem of completing the quantization of the Quispel mappings and obtaining the expressions for the quantum discrete Painlevé equations. As we will show, for the latter, the integrability constraints at the deautonomization level are so strong that they suffice in order to ensure integrability even after real-space quantization. However, another problem appears at a more fundamental level: that of the choice of the quantization prescription. In fact, the quantization rule must be consistent with the equations of motion and for most cases the Heisenberg prescription, $[x, y] = \mathrm{i}\hbar$, turns out to be inadequate. This problem will be the object of the next section.

2. The choice of the quantization

The type of quantization everybody is familiar with is that of Hamiltonian systems. In this case one starts with a pair of conjugate variables, say p, q, the Poisson bracket of which is just $\{p, q\} = 1$, and then chooses a quantization scheme among all those that lead to operators p, q with commutator $[p, q] = \mathrm{i}\hbar$. The Weyl prescription is perhaps the most popular among these ordering rules although not the only one. In the context of mappings there is no *a priori* physical interpretation of the mapping variables. This means that we do not have the Poisson bracket that may serve as a guide, but on the other hand, we are not bound any more by Heisenberg-type commutation relations. In fact, the latter may prove inconsistent for most mappings. Let us illustrate this point for the Quispel mapping. Its general form reads

$$x' = \frac{f_1(y) - xf_2(y)}{f_2(y) - xf_3(y)} \tag{1a}$$

$$y' = \frac{g_1(x') - yg_2(x')}{g_2(x') - yg_3(x')}. \tag{1b}$$

The functions f and g are quartic polynomials involving 18 parameters α_i, β_i, γ_i, δ_i, ε_i, ζ_i, κ_i, λ_i, μ_i for $i = 1, 2$. The mapping ($1a$, b) is integrable in the sense that it possesses an invariant that can be obtained by

$$(\alpha_1 + K\alpha_2)x^2y^2 + (\beta_1 + K\beta_2)x^2y + (\gamma_1 + K\gamma_2)x^2 + (\delta_1 + K\delta_2)xy^2 + (\varepsilon_1 + K\varepsilon_2)xy \\ + (\zeta_1 + K\zeta_2)x + (\kappa_1 + K\kappa_2)y^2 + (\lambda_1 + K\lambda_2)y + (\mu_1 + K\mu_2) = 0. \tag{2}$$

The case $f_3 = g_3 = 0$ was quantized in [11] under the quantization rule $[x, y] = 1$. Let us show that this rule is inconsistent with the general mapping by studying the particular case $f_2 = g_2 = 0$. Indeed, assuming $[x, y] = 1$ at every iteration, we would expect $[y, x'] = 1$ at the next step. Using the explicit form of the mapping for $f_2 = g_2 = 0$ we find readily that this is impossible unless $x' = x$, an obviously absurd constraint.

Still a solution to the problem exists, provided one uses the techniques of non-commutative geometry [12]. In order to lend a physical meaning to the 'quantum line' [13] it has been suggested [14] that, at the scale of the Planck length, non-Archimedean geometry must be introduced. Thus the spacetime coordinates do not commute but, rather, obey Weyl-type commutation relations

$$x_i x_j = q x_j x_i. \tag{3}$$

In order to cover the case of mappings of type (1) we introduce the general commutation relation

$$xy = qyx + \lambda x + \mu y + \nu. \tag{4}$$

Through scaling this relation can be always reduced to

$$xy = qyx + \lambda(x + y) + \nu. \tag{5}$$

Two cases can be distinguished, $q = 1$ and $q \neq 1$. In the latter a translation of the variables can always reduce the commutation relation to either of the forms $xy = qyx + \nu$ or $xy = qyx + \lambda(x + y)$. 'Heisenberg-type' commutation is obtained for $q = 1$ and moreover $\lambda = 0$, while 'Weyl-type' commutation is obtained for $q \neq 1$ and $\nu = 0$. In fact all kinds of commutations (5) can be related to the discrete Painlevé equations. The analysis can be performed at the autonomous level since the n-dependence of the coefficients cannot influence the commutation properties of the mapping variables.

We use the Quispel map as a starting point since all the known discrete Painlevé equations belong to this family at the autonomous limit. Starting from (1) we rewrite it in a more symmetric form

$$x'f_3(y)x - x'f_2(y) - f_2(y)x + f_1(y) = 0 \tag{6a}$$

$$y'g_3(x')y - y'g_2(x') - g_2(x')y + g_1(x') = 0. \tag{6b}$$

Let us work with equation ($6a$). Multiplying by y on the left and right we apply the commutation relation (5) and its iterated $yx' = qx'y + \lambda(x' + y) + \nu$. Using the commutation relations we bring all x's to the left and all xs to the right. Subtracting, only terms linear either in x or x' remain. We thus obtain the following condition for the vanishing of the terms proportional to x and x':

$$\lambda f_2 + \nu f_3 + y[(q-1)f_2 + \lambda f_3] = 0. \tag{7}$$

A similar expression involving gs is obtained from (6b). Before making further use of the mapping consistency condition, let us specify the relations between the parameters of the Quispel map in order to obtain the discrete Painlevé equations. The latter are symmetric ($f=g$) mappings for staggered variables: $x_{2n-1} \equiv x$, $x_{2n} \equiv y$, $x_{2n+1} \equiv x'$, $x_{2n+2} \equiv y'$. As we have explained in [15] the Painlevé mappings $\mathrm{P_I}$ to $\mathrm{P_V}$ are obtained for the ratio $p = f_3/(yf_3 - f_2)$, coinciding with the term that multiplies $\dot{y}^2$, in the continuous equation. We thus have $p=0$, ($f_3=0$) for d-$\mathrm{P_I}$ and d-$\mathrm{P_{II}}$, $p = 1/y$, ($f_2 = 0$) for d-$\mathrm{P_{III}}$, $p = 1/2y$, ($f_2 = -yf_3$) for d-$\mathrm{P_{IV}}$, and $p = \frac{1}{2}[1/y + 1/(y-1)]$, ($f_2 = y/(2y-1)f_3$) for d-$\mathrm{P_V}$. Thus, for d-$\mathrm{P_I}$ and d-$\mathrm{P_{II}}$, the compatible quantization condition is of Heisenberg type, i.e. $q=1$, $\lambda=0$, $\nu \neq 0$. For d-$\mathrm{P_{III}}$ we have $q \neq 0$ and $\lambda = \nu = 0$. For d-$\mathrm{P_{IV}}$ and d-$\mathrm{P_V}$ we substitute the relation between f_2, f_3 in (7) and we find $q=0$, $\nu=0$ and $\lambda=$ free for d-$\mathrm{P_{IV}}$, and $q=$ free $\neq 1$, $\nu = 0$ but $\lambda = (1-q)/2$. Thus a consistent commutation relation exists for everyone of the known discrete Painlevé equations.

One remark must be made at this point concerning the coalescence cascade of the discrete Painlevé equations. As is well known, the continuous Painlevé equations are related through the appropriate limits of dependent and independent quantities in the scheme d-$\mathrm{P_V} \to \{$d-$\mathrm{P_{IV}}$, d-$\mathrm{P_{III}}\} \to$ d-$\mathrm{P_{II}} \to$ d-$\mathrm{P_I}$. Thus the 'lower' Painlevé equations can be deduced from the 'higher' ones. We have shown in [15] that analogous relations exist for the discrete Painlevé equations. The quantization must be compatible with the coalescence procedure, in order to be consistent. For example, we obtain d-$\mathrm{P_{II}}(X)$ from d-$\mathrm{P_{III}}(x)$ through the limit $x = 1 + \delta X$ for $\delta \to 0$. Starting from the commutation relation $xy = qyx$ for d-$\mathrm{P_{III}}$ and putting $q = 1 + \delta^2 \nu$ we find, at the limit $\delta \to 0$: $XY = YX + \nu$, i.e. precisely the commutation relation for d-$\mathrm{P_{II}}$. In the case of d-$\mathrm{P_{II}}(x)$ to d-$\mathrm{P_{II}}(X)$ we have the same relation $x = 1 + \delta X$. Starting from $xy = yx + \nu$ and putting $\nu = \delta^2 \mu$ we find $XY = YX + \mu$, as expected. Similarly from d-$\mathrm{P_{IV}}(x)$ we obtain d-$\mathrm{P_{II}}(X)$ by $x = 1 + \delta X$. Here the commutation reads $xy = yx + \lambda(x+y)$ and it suffices to take $\lambda = \delta^2 \nu$ in order to obtain the commutation for d-$\mathrm{P_{II}}$. The discrete $\mathrm{P_V}(x)$ reduces either to d-$\mathrm{P_{III}}(X)$ through $x = X/\delta$ or to d-$\mathrm{P_{IV}}(X)$ through $x = \delta X$. Starting with $xy = qyx + \frac{1}{2}(1-q)(x+y)$ we obtain in the first case $XY = qYX$ and in the second (by $q = 1 - \delta\lambda$): $XY = YX + \lambda(X+Y)$, i.e. the right commutation relations in each case. Thus the quantization is compatible with the coalescence reduction scheme of the discrete Painlevé equations.

3. The autonomous case: quantization of the Quispel mappings

Once the proper quantization rule is chosen, it remains for us to prove, in the autonomous case, that there exists a quantity which remains invariant under the evolution of the quantum mapping. This quantity coincides with the classical invariant at the classical limit, where x, y are c-numbers, but may contain, in principle, quantum corrections. The case of $f_3 = 0$ (autonomous d-$\mathrm{P_I}$ and d-$\mathrm{P_{II}}$) was treated in detail in [11]. It was shown there that such an invariant exists and, moreover, if one chooses the proper ordering of its terms the quantum corrections can be made to vanish. In what follows we will focus on the autonomous versions of d-$\mathrm{P_{III}}$, d-$\mathrm{P_{IV}}$ and d-$\mathrm{P_V}$.

In the case of the autonomous d-P_{III}, ($f_2=0$), the evolution of the mapping writes

$$x'f_3(y)=-f_1(y)x^{-1} \tag{8}$$

where classically $f_3(y)=\alpha y^2+\beta y+\gamma$ and $f_1(y)=\gamma y^2+\zeta y+\mu$. The commutation rule is $xy=qyx$ and $yx'=qx'y$. This means that, in general, for any function F of y we have $xF(y)=F(qy)x$ and $x'F(y)=F(\frac{y}{q})x'$. In order to simplify notations we will denote $F(qy)$ by $\bar{F}$ and $F(\frac{y}{q})$ by $\underline{F}$. The evolution equation can now be written $x'x\underline{f}_3=-f_1$ or $x'x=-f_1\underline{f}_3^{-1}\equiv W$. Let us now (guided by the classical result, equation (2)) look for an invariant of the form

$$Kyx'=A(x')^2+Bx'+C \tag{9}$$

where A, B, C are quadratic polynomials in y: $A=a_2y^2+a_1y+a_0$ and similarly for B, C. We rewrite (9) as

$$Kyx'=y^2S_2+yS_1+S_0 \tag{10}$$

where $S_i=a_ix'^2+b_ix'+c_i$. In order to prove that the invariance of (9) we multiply with $xW^{-1}x$ on the right and find, using (8)

$$Kxy=qAW+qx\underline{B}+qx^2\underline{\underline{C}}\underline{W}^{-1}. \tag{11}$$

This last expression should coincide with (10) when one replaces (x', y) by (y, x). We obtain thus:

$$qAW=S_0 \tag{12a}$$

$$q\underline{B}=S_1 \tag{12b}$$

$$q\underline{\underline{C}}\underline{W}^{-1}=S_2. \tag{12c}$$

Equation (12a) defines $W=S_0/qA$, while (12b) is satisfied provided $b_2=qa_1$, and $qb_0=c_1$. Finally (12c) introduces one further constraint $c_2=q^2a_0$. Thus from the nine parameters a_i, b_i, c_i, five are free, one (b_1) can be set to zero by a redefinition of the constant K, and the remaining three are expressed in terms of the previous constraints. Going back to the notation of (8) we remark that because of the q factor in (12a) W should write $W=(1/q)(\gamma y^2+\zeta y+\mu)(\alpha y^2+\beta y+\gamma)^{-1}$. Thus the quantum corrections, i.e. q-dependent terms, enter not only the invariant but the evolution equation itself. This situation is not unlike the one for continuous systems [16] where it has been remarked that in order to preserve integrability upon quantization one must introduce quantum corrections in *both* the equations of motion *and* the invariant.

In the case of autonomous d-P_{IV} we start from a classical evolution equation $x'f_3x-x'f_2-f_2x+f_1=0$ with $f_3=\alpha y^2+\beta y+\gamma, f_2=-yf_3$ and $f_1=\beta y^3+(\varepsilon-\gamma)y^2-\mu$, and invariant $\alpha x^2y^2+\beta(x^2y+y^2x)+\gamma(x^2+y^2)+\varepsilon xy+\mu=K(x+y)$. The commutation rule is here $xy=yx+\lambda(x+y)$ and $yx'=x'y+\lambda(x'+y)$. Guided from the commutation relation and the form of the invariant (2), we introduce the auxiliary variable $z\equiv x+y$ and $z'\equiv x'+y$. The commutation rules now become $zy=(y+\lambda)z$ and $z'y=(y-\lambda)z'$, and, in general, for any function F of y: $zF(y)=F(y+\lambda)z$ and $z'F(y)=F(y-\lambda)z'$. Here again we will use the notation $\bar{F}=F(y+\lambda)$ and $\underline{F}=F(y-\lambda)$. Introducing z and z' in the mapping we can rewrite it as

$$z'f_3z=(y^2f_3-f_1) \tag{13}$$

or equivalently

$$z'z=(y^2f_3-f_1)\underline{f}_3^{-1}\equiv W. \tag{14}$$

Thus the problem can be treated in a way parallel to the d-P_{III}. We start with an invariant:

$$\begin{aligned} Kz' &= y^2 S_2 + yS_1 + S_0 \\ &= Ax'^2 + Bx' + C \\ &= Az'^2 + (B - a(y+\underline{y}))z' + (Ay^2 - By + C). \end{aligned} \tag{15}$$

We multiply on the left with $zW^{-1}z$ and bring all the zs to the left. However before comparing with the right-hand side of the first line of (15) we must bring all xs to the left (using also $z^2 = x^2 + x(y+\underline{y}) + y\underline{y}$). We find thus:

$$(Ay^2 - By + C)\bar{W}^{-1} = \bar{\bar{S}}_2 \tag{16a}$$

$$\bar{S}_2(y+\bar{y}) + B - A(y+\underline{y}) = \bar{S}_1 \tag{16b}$$

$$\bar{S}_2 y\bar{y} + \bar{y}(B - A(y+\underline{y})) + \bar{A}\bar{W} = \bar{S}_0. \tag{16c}$$

From these equations we recover $f_3 = \bar{\bar{S}}_2$ and we reconstruct f_1.

The situation for the autonomous d-P_V is quite similar. Here we have $f_2 = yA(y)$, $f_3 = (2y-1)A(y)$ and the equation of motion reads: $2x'yAx - (x'+y)A(y+x) = B$ where B is a polynomial quartic in y. The commutation rules are $xy = qyx + \lambda(x+y)$ and $yx' = qx'y + \lambda(x'+y)$, where $\lambda = (1-q)/2$. Guided from these expressions and the form of the classical invariant (2) we introduce the auxiliary variables $z = (2y-1)x - y$ and $z' = x'(2y-1) - y$. The advantage of the latter is that the new commutation relations read $zy = (qy+\lambda)z$ and $yz' = z'(qy+\lambda)$, and thus for any function F we have $zF(y) = F(qy+\lambda)z$ and $F(y)z' = z'F(qy+\lambda)$. The evolution writes now $z'Cz = D$, where $C = A/(2y-1)$ and D is rational with quintic numerator and $2y-1$ as a denominator. The analysis follows the same steps as in the case of d-P_{IV} resulting again to quantum corrections in both the invariant and the evolution equations. We will not go into these (increasingly tedious) details here.

The conclusion of the preceding analysis is that, as far as the autonomous Quispel mapping is concerned, there exists a consistent quantization scheme for the mappings corresponding to the autonomous versions of d-P_I to d-P_V. Moreover the quantization preserves integrability, i.e. there exists an invariant relation that is conserved in the quantum case, at the price of the choice of an ordering and the introduction of purely quantum correction terms in both the invariant and the mapping. Still the question of the quantization of the general symmetric 12-parameter Quispel map remains open at this stage. Thus the quantization of the autonomous d-P_{VI} cannot be given by relations of the form (5) and more general rules must be devised.

4. The non-autonomous case: quantization of the discrete Painlevé equations

The quantization of the discrete Painlevé equations leads us to considering the non-autonomous case. Here no invariant quantity exists. On the other hand, the special integrability properties of the Painlevé equations reflect themselves in the fact that there exists a Lax pair for them:

$$h\frac{d\Phi_n}{dh} = L_n\Phi_n, \; \Phi_{n+1} = M_n\Phi_n. \tag{17}$$

The discrete Painlevé equation is then obtained from the compatibility condition:

$$h\frac{dM_n}{dh} = L_{n+1}M_n - M_nL_n. \tag{18}$$

In the quantized case, the ordering is important and must be respected throughout. Let us illustrate this point in the case of d-P_I. (In what follows we will use the notation $\bar{x} \equiv x_{n+1}$, $x \equiv x_n$ and $\underset{\sim}{x} \equiv x_{n-1}$).

In [8] we have presented a 3×3 matrix realization of the Lax pairs

$$L = \begin{pmatrix} \lambda_1 & x & 1 \\ h & \lambda_2 & c - x - \underset{\sim}{x} \\ h\underset{\sim}{x} & h & \lambda_3 \end{pmatrix} \qquad M = \begin{pmatrix} (\lambda_1 - \lambda_2)x^{-1} & 1 & 0 \\ 0 & 0 & 1 \\ h & 0 & 0 \end{pmatrix} \tag{19}$$

where $\lambda_1 = \text{constant}$, $\lambda_2 = \lambda + n/2$, $\lambda_2 = \lambda + \frac{1}{2}(n+1)$. Using (18) one finds

$$\bar{x} + x + \underset{\sim}{x} = c + (\lambda_1 - \lambda_2)x^{-1} \tag{20}$$

i.e. the usual form of d-P_I without any quantum corrections. Moreover the expressions for L and M are the straightforward transcriptions of the classical ones: no ordering ambiguity appears. That this need not be always the case can be seen in the 2×2 Lax pair of Fokas *et al* [4] for the same equation d-P_I. Starting from:

$$M = \begin{pmatrix} 2\mu\bar{x}^{-1/2} & -\bar{x}^{-1/2}x^{1/2} \\ 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} -\mu(2x-c) & (x+\bar{x}-c)x^{1/2} \\ -(x+\underset{\sim}{x}-c)x^{1/2} & \mu(2x-c) \end{pmatrix} \tag{21}$$

and the compatibility condition: $\mathrm{d}M_n/\mathrm{d}\mu = L_{n+1}M_n - M_nL_n$ (note the different definition of the spectral parameter!), one has to make specific choices for the order of the x, $\underset{\sim}{x}$, $\bar{x}$ terms. The order given in (21) is in fact the one leading to the correct d-P_I. Indeed one finds, through the compatibility:

$$X - \bar{X} + 1 = \bar{x}^{1/2}x\bar{x}^{1/2} - x^{1/2}\bar{x}x^{1/2} \tag{22}$$

where $X = -4\mu^2 x + x^{1/2}(\bar{x} + x + \underset{\sim}{x} - c)x^{1/2}$. Now, it can be shown that if $[x, \bar{x}] = 1$, then the right-hand side of (22) vanishes, whereupon the latter is integrated to $X = n + \kappa$. Multiplying with $x^{-1/2}$ from right and left we obtain (20).

While in the last case the choice of ordering was important, for the remaining cases of known Lax pairs for d-Ps the situation is quite simple. No ordering ambiguities exist and one obtains the quantum analogue of the corresponding discrete Painlevé equation in a straightforward way. Thus for the second d-P_I we have

$$L = \begin{pmatrix} hx + \lambda_1 & h + y \\ h^2 + h(c_2 - y + c_1x - x^2) & h(c_1 - x) + \lambda_2 \end{pmatrix} \qquad M = \begin{pmatrix} (\lambda_1 - \lambda_2)y^{-1} & 1 \\ h & 0 \end{pmatrix} \tag{23}$$

where $\lambda_1 = \text{constant}$ and $\lambda_2 = n + c$, leading to:

$$x + \bar{x} = c + (\lambda_1 - \lambda_2)y^{-1} \tag{24a}$$

$$y + \underset{\sim}{y} = c_2 + xc_1 - x^2. \tag{24b}$$

Similarly, for d-P_I one finds:

$$\begin{aligned} L &= \begin{pmatrix} \lambda_1 & x & 1 & 0 \\ 0 & \lambda_2 & c - \underset{\sim}{x} & 1 \\ h & 0 & \lambda_3 & c - x \\ h\underset{\sim}{x} & h & 0 & \lambda_4 \end{pmatrix} \\ M &= \begin{pmatrix} (\lambda_1 - \lambda_2)x^{-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (\lambda_3 - \lambda_4)(c - x)^{-1} & 1 \\ h & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{25}$$

where $\lambda_1=$ constant, $\lambda_3=$ constant, $\lambda_2=(n-1)/2+\lambda$ and $\lambda_4=n/2+\lambda$ leading to

$$\bar{x}+\underline{x}=c+(\lambda_1-\lambda_2)x^{-1}+(\lambda_3-\lambda_4)(c-x)^{-1}. \tag{26}$$

Finally for d-$\mathrm{P_{III}}$ we recall that the isospectral problem is of q-difference type rather than a differential one

$$\Phi_n(\rho h)=L_n(h)\Phi_n(h), \Phi_{n+1}(h)=M_n(h)\Phi_n(h). \tag{27}$$

The theory of such q-difference equations has been developed since beginning of the century, cf, for example, [17, 18]. Recently, q-holonomic systems of q-difference equations have been seen to arise in connection with the quantum Yang–Baxter equations [19, 20]. A compatible system of a q-difference equation and a discrete-time evolution as a Lax pair for a discrete Painlevé equation was first derived in [8]. The compatibility, in the case of (27), reads

$$M_n(\rho h)L_n(h)=L_{n+1}(h)M_n(h). \tag{28}$$

Here we have, (recall $x\bar{x}=q\bar{x}x$):

$$L=\begin{pmatrix} \lambda_1 & \lambda_1+\kappa x^{-1} & \kappa x^{-1} & 0 \\ 0 & \lambda_2 & \lambda_2+\underline{x} & \underline{x} \\ hx & 0 & \lambda_3 & \lambda_3+x \\ h(\lambda_4+\alpha\kappa x^{-1}) & h\alpha\kappa x^{-1} & 0 & \lambda_4 \end{pmatrix}$$
$$M=\begin{pmatrix} \Lambda_1(\lambda_4+\alpha\kappa x^{-1})^{-1} & (\lambda_1 x+\kappa)(\lambda_2 x+\kappa)^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \Lambda_2(x+\rho\lambda_2)^{-1} & (x+\lambda_3)(x+\lambda_4)^{-1} \\ h & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

with $\alpha^2=\rho$, $\lambda_1=$ constant, $\lambda_3=$ constant, $\lambda_2=\lambda\alpha^{n-1}$, $\lambda_4=\lambda\alpha^n$, $\Lambda_1=\alpha\lambda_1-\lambda_4$, $\Lambda_2=\lambda_3-\rho\lambda_2$, $\kappa=C\alpha^n$.

The quantized mapping is obtained through the application of (28). It reads

$$\bar{x}(\kappa+\lambda_1 x)(\kappa+\lambda_2 x)^{-1}\underline{x}=\alpha\kappa(x+\lambda_3)(x+\lambda_4)^{-1}. \tag{30}$$

We can remark at this point that (30) does not depend explicitly on the quantum parameter q. This should be related to the choice of a specific factorization of f_3, f_1 via (8) so as to obtain (30). Had we written (30) in the form $\bar{x}\underline{x}=W(x)$, explicit q-dependence would have appeared.

5. Conclusions

In this paper we have addressed the question of quantization of autonomous mappings and of the (non-autonomous) discrete Painlevé equations. The conceptual difficulty that is encountered at the very first stage of this work is that of the proper choice of the quantization rule. We have shown that for each type of the known Painlevé equations, we can obtain a consistent quantization scheme. The latter is inspired by recently introduced techniques related to the quantum groups. In the autonomous case we have shown that it was possible to define a mapping and an invariant (both incorporating quantum corrections) in such a way that the latter remains indeed invariant under the iterations of the former. We must stress, however, at this point, that this quantization procedure is not consistent with the most general Quispel map (which would be the autonomous limit of the discrete Painlevé equation P_{VI}). More general quantization rules could be needed for the latter. In the non-autonomous case we have shown that the quantization is compatible with the Lax pair that ensures the linearization of each of the equations d-P_I, d-P_{II} and d-P_{III}. An interesting open problem is whether this holds also for d-P_{IV} and d-P_V. We believe that once their Lax pairs are obtained in the commutative case, the extension to the non-commutative one will not present undue difficulties. Another interesting direction of research would be the extension of our approach to the multi-dimensional case, investigating the effect of the quantization approach on integrable lattices.

Acknowledgments

The authors acknowledge interesting discussions with J Satsuma, J L Gervais, J Hietarinta and H Capel. They are particularly indebted to R Quispel, since the present work was the natural sequel to an earlier work of his in collaboration with one of us (FN).

References

[1] Nijhoff F W, Papageorgiou V G and Capel H W 1993 *Proc. Int. Workshop on Quantum Groups* (*Euler Int. Math. Institute, Leningrad*) ed L D Faddeev and P P Kulish (Berlin: Springer Verlag) and references therein
[2] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* **34D** 183
[3] Ramani A, Grammaticos B and Karra G 1992 *Physica* **181A** 115
[4] Its A R, Kitaev A V and Fokas A S 1990 *Usp. Matew. Nauk* **45** 6, 135
[5] Nijhoff F W and Papageorgiou V G 1991 *Phys. Lett.* **153A** 337
[6] Brézin E and Kazakov V A 1990 *Phys. Lett.* **236B** 144
Periwal V and Shevitz D 1990 *Phys. Rev. Lett.* **64** 1326
[7] Grammaticos B, Ramani A and Papageorgiou V G 1991 *Phys. Rev. Lett.* **67** 1825
[8] Papageorgiou V G, Nijhoff F W, Grammaticos B and Ramani A 1992 *Phys. Lett.* **164A** 57
[9] Novikov S P 1991 *Funct. Anal. Appl.* **24** 296
[10] Nijhoff F W, Capel H W and Papageorgiou V G 1992 *Phys. Rev.* A **46** 2155
Nijhoff F W and Capel H W 1992 *Phys. Lett.* **163A** 49
[11] Quispel G R W and Nijhoff F W 1992 *Phys. Lett.* **161A** 419
[12] Connes A 1986 *Publ. Math. IHES* **62**
[13] Manin Yu I 1989 Quantum groups and noncommutative geometry *Publ. CRM* University de Montreal
Woronowicz S 1987 *Commun. Math. Phys.* **111** 613; 1989 *Commun. Math. Phys.* **122** 125
[14] Volovich I 1987 *Class. Quantum Grav.* **4** L83
Aref'eva I Ya and Volovich I V 1991 *CERN Preprint* TH.6137/91
[15] Ramani A, Grammaticos B and Hietarinta J 1991 *Phys. Rev. Lett.* **67** 1829
[16] Hietarinta J 1984 *J. Math. Phys.* **25** 1833
[17] Birkhoff G D 1913 *Proc. Am. Acad. Arts Sci.* **49** 521
[18] Trjitzinsky W J 1933 *Acta Math.* **61** 1
[19] Frenkel T B and Reshetikin N Yu 1993 Quantum affine algebras and holonomic difference equations *Proc. Int. Conf. on Differential Geometric Methods in Theoretical Physics* to appear
[20] Aomoto K 1988 A note on holonomic q-difference systems *Algebraic Analysis* vol 1, ed M Kashiwara and T Kawai (Boston, MA: Academic) p 25